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YANG-MILLS THEORY WITH THE PONTRYAGIN TERM ON MANIFOLDS WITH A BOUNDARY

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Abstract

The $3 + 1$ dimensional Yang-Mills theory with the Pontryagin term included is studied on manifolds with a boundary. Based on the geometry of the universal bundle for Yang-Mills theory, the symplectic structure of this model is exhibited. The topological type of the quantization line bundles is shown to be determined by the torsion elements in the cohomology of the gauge orbit space.

1. Introduction

Gauge theories whose actions combine an ordinary Yang-Mills kinetic term and a topological term possess an interesting mathematical structure. Prominent examples are the Yang-Mills theory in odd dimensions including the Chern-Simons term [1,2] -afterwards called CSYM theory- and secondly the Yang-Mills theory in even dimensions with the Pontryagin term added to which we refer as PYM theory [3]. These topological terms introduce functional abelian background fields in the corresponding configuration space, whose geometrical structure can be traced back to cohomological properties of the Yang-Mills orbit space \mathcal{M} [4-6]: Depending on the space dimension and the gauge group, a consistent quantization of the CSYM theory imposes a quantization condition on the coupling parameter which has its origin in the second integer cohomology group of \mathcal{M} [5,6]. On manifolds with a nontrivial boundary the generators of gauge transformations satisfy an anomalous commutator algebra which is cohomologically equivalent to the Faddeev-Mickelsson current algebra [2,7]. Recently, the discussion of pure Chern-Simons theory on manifolds with boundaries has lead to a detailed investigation on so-called edge states [8], which carry representations of the Kac-Moody algebra.

On the other hand, it is well known [3] that a Pontryagin term (θ -term) can be added in nonabelian gauge theories due to instanton effects. This gives rise to a kind of Aharonov Bohm effect where the θ parameter is identified with the magnetic flux associated with a vortex structure in the gauge orbit space [4]. Topologically, this effect is related to the first integer cohomology group of \mathcal{M} [6].

In this paper we consider the PYM-theory on manifolds with a nontrivial boundary. We restrict ourselves to the case of 4 dimensional manifolds but the generalization to higher dimensions is straightforward. Since we aim at a Hamiltonian description, we assume that the 4-manifold N is of the form $N = \mathbb{R} \times M$ and that M has a nontrivial boundary ∂M . The purpose of this paper is to clarify the symplectic structure of the classical phase space and to study the corresponding quantum theory of this model in terms of the geometry of the underlying gauge orbit space. The starting point is the classical action

$$S = \int_{\mathbb{R} \times M} \text{tr}(F_{\bar{A}} \wedge \bar{*}F_{\bar{A}}) + \frac{\theta}{8\pi^2} \int_{\mathbb{R} \times M} \text{tr}(F_{\bar{A}} \wedge F_{\bar{A}}), \quad (1)$$

where $F_{\bar{A}} = d\bar{A} + \frac{1}{2}[\bar{A}, \bar{A}]$ is the Yang-Mills field strength, \bar{A} is regarded as connection on a principal bundle \bar{P} over N with a compact, connected, simple Lie group G as structure group, and $\bar{*}$ is the Hodge operator induced by a fixed Riemannian structure on N .

It will be shown that the symplectic data of the action (1) are related to certain secondary characteristic classes of the universal bundle for Yang-Mills theory which originally has been introduced by Atiyah and Singer [9] in order to study nonabelian anomalies. From the mathematical viewpoint, the contribution of the boundary can be expressed in terms of a new set of descent equations in the universal bundle. These equations have previously been used in the analysis of covariant anomalies in Yang-Mills theory [10].

The classical reduced phase space is the cotangent bundle $T^*\mathcal{M}$ of the gauge orbit space equipped with a symplectic structure which differs from the canonical one by the pullback of a closed two form defined on the gauge orbit space associated with the Yang-Mills fields on the boundary manifold.

Using the framework of geometric quantization [11], it turns out that quantization of the PYM model is not unique, if the corresponding gauge group is disconnected. Topologically, the quantum line bundles are related to the torsion elements in the integer cohomology group of \mathcal{M} . As a consequence, no quantization condition must be imposed on the coupling parameter θ . The physical states are invariant under infinitesimal gauge transformations but in general they acquire a phase under gauge transformations not belonging to the identity component of the gauge group.

2. The geometrical setup

In this section we shall prepare the necessary mathematical constructions in order to study the classical and quantum theory of the model defined by (1). It will be shown in the next section that a symplectic formulation of this model is intimately related with the geometrical structure of the universal bundle for Yang-Mills theory.

Let M be a compact, connected 3-manifold with boundary $\Sigma := \partial M$ and let $P(M, G)$ be the principal fiber bundle with structure group G which is given by restricting \bar{P} to M . Associated to P with respect of the adjoint action of G on its Lie algebra \mathfrak{g} is the adjoint bundle $adP := P \times_G \mathfrak{g}$. The gauge group \mathcal{G} is defined to be the group of those vertical bundle automorphisms of P which act freely on the space \mathcal{A} of all connections on P . For $A \in \mathcal{A}$ and $u \in \mathcal{G}$ this action is given by the pull-back $A \mapsto u^*A$. Let $\mathcal{A}(\mathcal{M}, \pi_{\mathcal{A}}, \mathcal{G})$ be the corresponding principal \mathcal{G} bundle. The gauge algebra $Lie\mathcal{G}$ can be identified with $\Omega^0(M, adP)$ and the fundamental vector field on \mathcal{A} with respect to the given \mathcal{G} action is $Z_\xi = d_A\xi$ for $\xi \in Lie\mathcal{G}$. Here $d_A: \Omega^*(M, adP) \rightarrow \Omega^{*+1}(M, adP)$ is the covariant exterior derivative with respect to A . There is a natural scalar product on the space $\Omega(M, adP)$ of all adP -valued differential forms on M , given by

$$(\alpha, \beta) := \int_M tr(\alpha \wedge \star \beta), \quad \alpha, \beta \in \Omega^p(M, adP), \quad (2)$$

where \star is the corresponding Hodge operator, satisfying $\star^2 = (-1)^{p(3-p)}$.

Let $i: \partial M \hookrightarrow M$ be the inclusion of the boundary, then we denote the restriction of P (i.e. pullback i^*P) by $P_\Sigma := i^*P$ with induced map $\bar{i}: P_\Sigma \rightarrow P$. Let \mathcal{B} the space of all connections on P_Σ and let \mathcal{H} be the gauge group for $P_\Sigma \rightarrow \Sigma$. Evidently, \bar{i} induces a map $\hat{i}: \mathcal{A} \rightarrow \mathcal{B}$ by $\hat{i}(A) \equiv \hat{A} := \bar{i}^*A$ and it also induces a map $\mathcal{G} \rightarrow \mathcal{H}$. In the following we shall write $\hat{\phi} := \bar{i}^*\phi$ for the restriction of any $\phi \in \Omega^*(M, adP)$. If $\hat{\star}$ is the induced Hodge operator on Σ satisfying $\hat{\star}^2 = (-1)^{p(2-p)}$, an inner product on $\Omega^*(\Sigma, adP_\Sigma)$ is defined by $(\phi_1, \phi_2)_\Sigma = \int_\Sigma tr(\phi_1 \wedge \hat{\star}\phi_2)$ for $\phi_i \in \Omega^p(\Sigma, adP_\Sigma)$.

Following Atiyah and Singer [9], let us consider the principal G bundle $\mathcal{B} \times P_\Sigma \rightarrow \mathcal{B} \times \Sigma$, which admits a natural \mathcal{H} action. If the quotient is taken along this action, one obtains a principal G bundle (the so called universal bundle) $\mathcal{B} \times_{\mathcal{H}} P_\Sigma \rightarrow \mathcal{N} \times \Sigma$, where $\mathcal{N} := \mathcal{B}/\mathcal{H}$ denotes the corresponding orbit space. There is a natural connection in the principal \mathcal{H} bundle $\mathcal{B}(\mathcal{N}, \mathcal{H})$, namely

$$\alpha_B(\rho_B) = \hat{G}_B \hat{d}_B^* \rho_B, \quad \rho_B \in T_B \mathcal{B} \cong \Omega^1(\Sigma, adP_\Sigma). \quad (3)$$

Here $\hat{G}_B = (\hat{d}_B^* \hat{d}_B)^{-1}$ is the Green operator and $\hat{d}_B^* = -\hat{\star} \hat{d}_B \hat{\star}$ is the adjoint of the covariant exterior derivative \hat{d}_B on P_Σ . Let us consider the following connection

$$\omega_{(B,q)}(\rho_B, \zeta_q) = (\alpha_B(\rho_B))(q) + B_q(\zeta_q), \quad (\rho_B, \zeta_q) \in T_{(B,q)}(\mathcal{B} \times P_\Sigma) \quad (4)$$

on $\mathcal{B} \times P_\Sigma \rightarrow \mathcal{B} \times \Sigma$, which descends to a connection $\bar{\omega}$ on $\mathcal{B} \times_{\mathcal{H}} P_\Sigma \rightarrow \mathcal{N} \times \Sigma$. According to the bigraded structure of the space of differential forms on $\mathcal{B} \times P_\Sigma$, the curvature Ω_ω of ω is determined by the components

$$\begin{aligned}\Omega_{\omega}^{(2,0)}(B,q)(\rho_1, \rho_2) &= \mathcal{F}_B(\rho_1, \rho_2) = \hat{G}_B \hat{\star}([\rho_1^h, \hat{\star} \rho_2^h] - [\rho_2^h, \hat{\star} \rho_1^h]) \\ \Omega_{\omega}^{(1,1)}(B,q)(\rho, \zeta_q) &= \rho_q^h(\zeta_q) \\ \Omega_{\omega}^{(0,2)}(B,q)(\zeta_q^1, \zeta_q^2) &= F_B(\zeta_q^1, \zeta_q^2),\end{aligned}\tag{5}$$

where F_B is the curvature of B and $\rho_i^h = \rho_i - d_B \alpha_B(\rho_i)$, with $\rho_i \in T_B \mathcal{B}$, are the horizontal projections with respect to α .

Another natural connection on $\mathcal{B} \times P_\Sigma \rightarrow \mathcal{B} \times \Sigma$ is given by $\eta_{(B,q)}(\rho_B, \zeta_q) = B_q(\zeta_q)$, which, however, does not descend to the \mathcal{H} quotient. For $\rho_B \in T_B \mathcal{B}$ and $\zeta_q^i \in T_q P_\Sigma$ the components of its curvature Ω_η are given by

$$\Omega_\eta^{(2,0)} = 0, \quad \Omega_\eta^{(1,1)}(B,q)(\rho_B, \zeta_q) = (\rho_B)_q(\zeta_q), \quad \Omega_\eta^{(0,2)}(B,q)(\zeta_q^1, \zeta_q^2) = F_B(\zeta_q^1, \zeta_q^2).\tag{6}$$

Analogously, we can define a connection φ in the principal G bundle $\mathcal{A} \times P \rightarrow \mathcal{A} \times M$ by $\varphi_{(A,p)}(\tau_A, X_p) := A_p(X_p)$, whose curvature Ω_φ has the following components

$$\Omega_\varphi^{(2,0)} = 0, \quad \Omega_\varphi^{(1,1)}(A,p)(\tau_A, X_p) = (\tau_A)_p(X_p), \quad \Omega_\varphi^{(0,2)}(A,p)(X_p^1, X_p^2) = F_A(X_p^1, X_p^2),\tag{7}$$

where $\tau_A \in T_A \mathcal{A}$ and $X_p^i \in T_p P$. It is evident that $(id \times \bar{i})^* \varphi = (\hat{i} \times id)^* \eta$ holds.

In the remainder of this paper we shall consider the trace as an ad-invariant polynomial Q on \mathfrak{g} of degree 2. Let $\mathcal{P} \rightarrow X$ be any of the principal bundles introduced before and let α be a corresponding connection on \mathcal{P} with curvature F . Then the exact 4-form $Q(F) := Q(F, F)$ on \mathcal{P} descends to a well defined form on X . If α_1 and α_2 are two connections on \mathcal{P} with curvatures F_1 and F_2 , then the secondary characteristic 3-form $TQ(\alpha_1, \alpha_2) \in \Omega^3(X, \mathbb{R})$ satisfies [12]

$$Q(F_1) - Q(F_2) = d_X TQ(\alpha_1, \alpha_2),\tag{8}$$

where $TQ(\alpha_1, \alpha_2) = 2 \int_0^1 dt Q(\alpha_1 - \alpha_2, \mathcal{F}_t)$ and \mathcal{F}_t is the curvature of the interpolating connection $(1-t)\alpha_2 + t\alpha_1$.

If G is not simply connected, then the bundles P and P_Σ may be nontrivial. In that case we choose a fixed background field $a \in \mathcal{A}$ which is extended to a connection in $\mathcal{A} \times P$ in a natural way (we shall denote it with the same symbol) and has curvature Ω_a . Because of dimensional reasons, $Q(\Omega_a) = 0$. Let \hat{a} denote the restriction of this connection to $\mathcal{A} \times P_\Sigma$.

The Chern-Weil formula (8) yields the following set of descent equations for the

connections ω , η , φ and a

$$d_{\mathcal{B}}Q(\Omega_{\omega})^{(k,4-k)} + (-1)^{k+1}d_{\Sigma}Q(\Omega_{\omega})^{(k+1,3-k)} = 0 \quad (9a)$$

$$d_{\mathcal{B}}Q(\Omega_{\eta})^{(k,4-k)} + (-1)^{k+1}d_{\Sigma}Q(\Omega_{\eta})^{(k+1,3-k)} = 0 \quad (9b)$$

$$d_{\mathcal{A}}Q(\Omega_{\varphi})^{(k,4-k)} + (-1)^{k+1}d_MQ(\Omega_{\varphi})^{(k+1,3-k)} = 0 \quad (9c)$$

$$Q(\Omega_{\omega})^{(k,4-k)} = d_{\mathcal{B}}TQ(\omega, \hat{a})^{(k-1,4-k)} + (-1)^k d_{\Sigma}TQ(\omega, \hat{a})^{(k,3-k)} \quad (9d)$$

$$Q(\Omega_{\eta})^{(k,4-k)} = d_{\mathcal{B}}TQ(\eta, \hat{a})^{(k-1,4-k)} + (-1)^k d_{\Sigma}TQ(\eta, \hat{a})^{(k,3-k)} \quad (9e)$$

$$Q(\Omega_{\varphi})^{(k,4-k)} = d_{\mathcal{A}}TQ(\varphi, a)^{(k-1,4-k)} + (-1)^k d_MTQ(\varphi, a)^{(k,3-k)} \quad (9f)$$

$$(Q(\Omega_{\omega}) - Q(\Omega_{\eta}))^{(k,4-k)} = d_{\mathcal{B}}TQ(\omega, \eta)^{(k-1,4-k)} + (-1)^k d_{\Sigma}TQ(\omega, \eta)^{(k,3-k)}, \quad (9g)$$

where $d_{\mathcal{A}}$, $d_{\mathcal{B}}$, d_M , d_{Σ} are the corresponding exterior derivatives and the superscripts indicate the form degrees with respect to the bigraded structure of the algebra of differential forms on $\mathcal{A} \times M$ and $\mathcal{B} \times \Sigma$, respectively. Finally, the secondary characteristic forms satisfy the following identity

$$TQ(\omega, \eta) = TQ(\omega, \hat{a}) - TQ(\eta, \hat{a}) + d_{\mathcal{A} \times \Sigma} SQ(\omega, \eta, \hat{a}), \quad (10)$$

for the 2-form $SQ(\omega, \eta, \hat{a}) = Q(\omega - \eta, \eta - \hat{a}) \in \Omega^2(\mathcal{B} \times \Sigma, \mathbb{R})$. An application of these descent equations for the determination of covariant Yang-Mills anomalies has been discussed in [10].

3. The classical phase space of the PYM theory

In this section we want to analyze the classical structure of the model defined by the action (1). We leave both the values of the gauge fields and the gauge transformations free on the boundary. Since we want to study the system in a fixed time formalism, we introduce the space $\mathcal{A}^0 = \Omega^0(M, adP)$ of all scalar potentials. The classical configuration space is $\mathcal{A} \times \mathcal{A}^0$ and the Lagrangian associated to (1) is the real valued function

$$L(A, A_0, \dot{A}, \dot{A}_0) = \frac{1}{2} \|\dot{A} - d_A A_0\|^2 - \frac{1}{2} \|F_A\|^2 + \frac{\theta}{8\pi^2} (\dot{A} - d_A A_0, \star F_A), \quad (11)$$

where $\|\cdot\|$ is the norm associated with (2) and $(\dot{A}, \dot{A}_0) \in \Omega^1(M, adP) \times \Omega^0(M, adP)$ are the fiber coordinates in the tangent bundle $T(\mathcal{A} \times \mathcal{A}^0)$.

Elements of the phase space are pairs (A, A_0, Π, Π_0) where, according to the inner product (2), (Π, Π_0) is regarded as an element of $\Omega^1(M, adP) \times \Omega^0(M, adP)$. The conjugate momenta are given by

$$\Pi_0 = 0, \quad \Pi = \frac{\delta L}{\delta A} = \dot{A} - d_A A_0 + \frac{\theta}{8\pi^2} \star F_A. \quad (12)$$

Hence the corresponding Hamilton function reads

$$H(A, A_0, \Pi, \Pi_0) = \frac{1}{2} \|\Pi - \frac{\theta}{8\pi^2} \star F_A\|^2 + \frac{1}{2} \|F_A\|^2 + (\Pi, d_A A_0). \quad (13)$$

The Poisson bracket $\{, \}$ is induced by the canonical symplectic two form on $T^*(\mathcal{A} \times \mathcal{A}_0)$

$$\begin{aligned} \mathfrak{K}_{\mathcal{A}}((\tau_1, \tau_1^0, \sigma_1, \sigma_1^0), (\tau_2, \tau_2^0, \sigma_2, \sigma_2^0)) \\ = \int_M (Q(\tau_2, \star \sigma_1) + Q(\tau_2^0, \star \sigma_1^0) - Q(\tau_1, \star \sigma_2) - Q(\sigma_1^0, \star \tau_2^0)), \end{aligned} \quad (14)$$

with $(\tau_i, \tau_i^0, \sigma_i, \sigma_i^0) \in T(T^*(\mathcal{A} \times \mathcal{A}_0))$, $i = 1, 2$. The Lagrangian is singular and gives rise to the primary constraint $\Pi_0 = 0$. The secondary constraint is given by

$$J_{\xi} = \{H, \Pi_0(\xi)\} = (\Pi, d_A \xi), \quad \xi \in \text{Lie} \mathcal{G} \quad (15)$$

which is of first class $\{J_{\xi}, J_{\eta}\} = J_{[\xi, \eta]}$, and there are no constraints of higher order, since $\{J_{\xi}, H\} = J_{[A_0, \xi]}$. In the following we eliminate the constraint $\Pi_0 = 0$ by fixing $A_0 = 0$.

We want to note that the constraint algebra of the PYM model is of first class whereas it is of second class in the CSYM theory on a manifold with boundary. In the latter only gauge transformations which reduce to the identity at the boundary can be regarded as gauge degrees of freedom (see Ref. 2 and references therein).

In order to identify the reduced classical phase space of our model, we note that the Hamiltonian (13) describes the motion of the Yang-Mills field in the background of the abelian functional field $\frac{\theta}{8\pi^2} \star F_A$. Instead of analyzing the theory on $T^* \mathcal{A}$ with Hamiltonian (13) and symplectic structure (14), we follow the method proposed by Sternberg [13] to rewrite this system in terms of the new momenta $\Pi - \frac{\theta}{8\pi^2} \star F_A$. In consequence, this modifies the canonical symplectic structure. Therefore, let us consider the following diffeomorphism γ along the fibers of the cotangent bundle $T^* \mathcal{A} \xrightarrow{\pi_T} \mathcal{A}$

$$\gamma(\Pi)_A(\tau) := (\Pi, \tau) + \frac{\theta}{8\pi^2}(\tau, \star F_A) = (\Pi, \tau) + \frac{\theta}{16\pi^2} \int_M Q(\Omega_{\varphi})^{(1,3)}(\tau), \quad (16)$$

where $\Pi \in T_A^* \mathcal{A}$ and $\tau \in T_A \mathcal{A}$. For the identification of the last term on the right hand side of (16) we have used (7).

Setting $\beta := \frac{\theta}{16\pi^2} \int_M Q(\Omega_{\varphi})^{(1,3)}$ and using (9c), the new symplectic structure which includes the interaction with the functional background field reads

$$\mathfrak{K}_{\beta} := \gamma^* \mathfrak{K}_{\mathcal{A}} = \mathfrak{K}_{\mathcal{A}} + \pi_T^* R_{\beta}, \quad (17)$$

where the two form $R_{\beta} := d_A \beta \in \Omega^2(\mathcal{A}, \mathbb{R})$ is given by

$$R_{\beta}(\tau_1, \tau_2) = -\frac{\theta}{16\pi^2} \hat{i}^* \int_{\Sigma} Q(\Omega_{\eta})^{(2,2)}(\tau_1, \tau_2) = \frac{\theta}{8\pi^2} \int_{\Sigma} Q(\hat{\tau}_1, \hat{\tau}_2), \quad \tau_i \in T_A \mathcal{A}. \quad (18)$$

Finally, the new Hamiltonian has the form

$$\tilde{H} = \frac{1}{2} \|\Pi\|^2 + \frac{1}{2} \|F_A\|^2 + (\Pi + \frac{\theta}{8\pi^2} \star F_A, d_A A_0), \quad (19)$$

and therefore

$$\tilde{J}_{\xi}(A, \Pi) = (\Pi, d_A \xi) + \frac{\theta}{8\pi^2} \int_{\Sigma} Q(\hat{\xi}, F_{\hat{A}}) \quad (20)$$

is the corresponding Gauss-law constraint. Note that the second term in (20) coincides with the expression for the two dimensional covariant anomaly [10] on Σ .

There is a natural symplectic action of the gauge group \mathcal{G} on $(T^*\mathcal{A}, \mathfrak{K}_\beta)$ with infinitesimal generator $X_\xi \in \mathfrak{X}(T^*\mathcal{A})$,

$$X_\xi(\Phi)(A, \Pi) = \frac{d}{dt}\big|_{t=0} (\Phi(A + td_A\xi, \Pi) - \Phi(A, \Pi + t[\xi, \Pi])), \quad \Phi \in C^\infty(T^*\mathcal{A}), \quad (21)$$

satisfying $i_{X_\xi}\mathfrak{K}_\beta = -d_{T^*\mathcal{A}}\tilde{J}_\xi$.

The classical PYM model is described by the constrained system $(T^*\mathcal{A}, \mathfrak{K}_\beta, \tilde{J})$ consisting of the symplectic manifold $(T^*\mathcal{A}, \mathfrak{K}_\beta)$, the Gauss constraint \tilde{J} which is viewed as an equivariant momentum map $T^*\mathcal{A} \rightarrow (\text{Lie}\mathcal{G})^*$ from the phase space to the dual of the gauge algebra and the Hamiltonian \tilde{H} . Via the Marsden Weinstein reduction [14], the true phase space of the model is obtained as the quotient $\tilde{J}^{-1}(0)/\mathcal{G}$ and the symplectic form $\overline{\mathfrak{K}_\beta}$ is given by restricting \mathfrak{K}_β to $\tilde{J}^{-1}(0)$ and projecting onto the orbit space.

PROPOSITION 1. *Consider the constrained system $(T^*\mathcal{A}, \mathfrak{K}_\beta, \tilde{J})$. There exists a symplectomorphism between the symplectic manifolds $(\tilde{J}^{-1}(0)/\mathcal{G}, \overline{\mathfrak{K}_\beta})$ and $(T^*\mathcal{M}, \overline{\mathfrak{K}_\omega})$.* ■

PROOF. Let us consider the symplectic \mathcal{G} space $(T^*\mathcal{A}, \mathfrak{K}_\omega, J)$, with symplectic two form $\mathfrak{K}_\omega = \mathfrak{K}_\mathcal{A} + \pi_T^*R_\omega$, where

$$R_\omega(\tau_1, \tau_2) = -\frac{\theta}{16\pi^2}\hat{i}^* \int_\Sigma Q(\Omega_\omega)^{(2,2)}(\tau_1, \tau_2) = \frac{\theta}{8\pi^2} \int_\Sigma [Q(\hat{\tau}_1^h, \hat{\tau}_2^h) - Q(\mathcal{F}_\mathcal{A}(\hat{\tau}_1, \hat{\tau}_2), F_\mathcal{A})], \quad (22)$$

and $J_\xi = (\Pi, d_A\xi)$ (15) is the equivariant momentum satisfying $i_{X_\xi}\mathfrak{K}_\omega = -d_{T^*\mathcal{A}}J_\xi$.

The two form R_ω descends to $\overline{R_\omega}$, which is obtained by pulling back $-\frac{\theta}{16\pi^2} \int_\Sigma Q(\Omega_\omega)^{(2,2)}$ along the induced map $\mathcal{M} \rightarrow \mathcal{N}$. Let $\bar{\pi}_T: T^*\mathcal{M} \rightarrow \mathcal{M}$ be the projection and let $\mathfrak{K}_\mathcal{M}$ be the canonical symplectic form on $T^*\mathcal{M}$ then the symplectic reduction of $(T^*\mathcal{A}, \mathfrak{K}_\beta, \tilde{J})$ yields $(T^*\mathcal{M}, \overline{\mathfrak{K}_\omega})$, where $\overline{\mathfrak{K}_\omega} = \mathfrak{K}_\mathcal{M} + \bar{\pi}_T^*\overline{R_\omega}$.

Let $j: J^{-1}(0) \hookrightarrow T^*\mathcal{A}$ and $\tilde{j}: \tilde{J}^{-1}(0) \hookrightarrow T^*\mathcal{A}$ be the inclusions, then we consider the following diffeomorphism along the fibers of $T^*\mathcal{A} \rightarrow \mathcal{A}$

$$\chi(\Pi)_A(\tau) := (\Pi, \tau) - \frac{\theta}{16\pi^2} \int_\Sigma TQ(\omega, \eta)^{(1,2)}(\hat{\tau}) = (\Pi, \tau) - \frac{\theta}{8\pi^2} (\hat{d}_\mathcal{A} \hat{G}_\mathcal{A} \hat{\star} F_\mathcal{A}, \hat{\tau})_\Sigma, \quad (23)$$

where $\tau \in T_A\mathcal{A}$ and the explicit expression for $TQ(\omega, \eta)$ was calculated using (5) and (6).

Because of (9g) one obtains $\mathfrak{K}_\omega = \chi^*\mathfrak{K}_\beta$, so that χ induces a diffeomorphism $J^{-1}(0) \rightarrow \tilde{J}^{-1}(0)$, which also is \mathcal{G} equivariant. Finally, the relation $\tilde{j} \circ \chi = \chi \circ j$ implies that χ is a presymplectomorphism. □

In the new coordinates, the dynamical structure of the model is thus governed by the Hamiltonian

$$H_\theta = \frac{1}{2}\|\Pi\|^2 - \frac{\theta}{8\pi^2}(\hat{\Pi}, \hat{d}_\mathcal{A} \hat{G}_\mathcal{A} \hat{\star} F_\mathcal{A})_\Sigma + \frac{1}{2}\|F_\mathcal{A}\|^2 + (\Pi, d_A A_0) + \frac{1}{2}\left(\frac{\theta}{8\pi^2}\right)^2 \|\hat{d}_\mathcal{A} \hat{G}_\mathcal{A} \hat{\star} F_\mathcal{A}\|_\Sigma^2, \quad (24)$$

which descends to a Hamiltonian \bar{H}_θ on the symplectic quotient $T^*\mathcal{M}$. We should remark that the symplectomorphism between $\tilde{J}^{-1}(0)/\mathcal{G}$ and $T^*\mathcal{M}$ requires the introduction of a connection in $\mathcal{B}(\mathcal{N}, \mathcal{H})$. However, with our choice (3), the symplectic data of the PYM model are similar to those of the CSYM theory [5]. This is a consequence of the fact that the Pontryagin density is the derivative of the Chern-Simons term.

4. Quantization of the PYM theory

In this section we want to study the quantum theory of the PYM model in the framework of geometric quantization [10]. Essentially, one has to introduce a prequantum line bundle and a polarization of the phase space. We consider the extended phase space quantization of $(T^*\mathcal{A}, \mathfrak{K}_\omega, J)$ following Dirac [15]. The idea is to quantize the unconstrained system and then to impose the constraints as conditions on the states.

Let $\mathcal{L}'_0 = T^*\mathcal{A} \times \mathbb{C}$ be the trivial prequantum line bundle over $T^*\mathcal{A}$ with connection $\nabla := d_{T^*\mathcal{A}} - i\vartheta - i\varepsilon$, where

$$\varepsilon = -\frac{\theta}{16\pi^2} \hat{i}^* \int_{\Sigma} TQ(\omega, \hat{a})^{(1,2)} \in \Omega^1(\mathcal{A}, \mathbb{R}). \quad (25)$$

It is evident from (9d) and (22) that $d_{\mathcal{A}}\varepsilon = R_\omega$. Associated with any observable $\Phi \in C^\infty(T^*\mathcal{A})$ is the first order differential operator $\mathcal{O}_\Phi := -i\nabla_{X_\Phi} + \Phi$, where X_Φ is defined by $i_{X_\Phi}\mathfrak{K}_\omega = -d_{T^*\mathcal{A}}\Phi$. Let \mathcal{O}_ξ denote the Gauss constraint operator associated with the momentum map (15) $J_\xi \in C^\infty(T^*\mathcal{A})$, then the physical admissible states are those sections of \mathcal{L}'_0 which are annihilated by \mathcal{O}_ξ and compatible with the chosen polarization. In the Schrödinger polarization of $T^*\mathcal{A}$, this requires the restriction to sections ψ of the trivial line bundle \mathcal{L}_0 on \mathcal{A} , satisfying

$$\nabla_{Z_\xi}^\varepsilon \psi(A) = L_{Z_\xi} \psi(A) + \frac{i\theta}{16\pi^2} (\hat{\xi}, \hat{\star}(F_{\hat{A}} + F_{\hat{a}} - \frac{1}{2}[\hat{A} - \hat{a}, \hat{A} - \hat{a}]))_\Sigma \psi(A) = 0, \quad \xi \in \text{Lie}\mathcal{G} \quad (26)$$

where $\nabla^\varepsilon := d_{\mathcal{A}} - i\varepsilon$ is the covariant derivative on \mathcal{L}_0 with curvature $-iR_\omega$ and L_{Z_ξ} is the Lie derivative on \mathcal{A} along the fundamental vector field Z_ξ . The conjugate momentum is represented by the operator $-i\nabla^\varepsilon$ and hence the Hamilton operator reads

$$H_\theta = -\frac{1}{2} \|\nabla^\varepsilon\|^2 + \frac{i\theta}{8\pi^2} (\hat{\nabla}^\varepsilon, \hat{d}_{\hat{A}} \hat{G}_{\hat{A}} \hat{\star} F_{\hat{A}})_\Sigma + \frac{1}{2} \|F_A\|^2 + \frac{1}{2} \left(\frac{\theta}{8\pi^2}\right)^2 \|\hat{d}_{\hat{A}} \hat{G}_{\hat{A}} \hat{\star} F_{\hat{A}}\|_\Sigma^2. \quad (27)$$

Notice that the second term on the right hand side of (27) is the expression for the operator \mathcal{O}_f , evaluated in the Schrödinger polarization, where $f := -\frac{\theta}{8\pi^2} (\hat{\Pi}, \hat{d}_{\hat{A}} \hat{G}_{\hat{A}} \hat{\star} F_{\hat{A}})_\Sigma$ in (24) is regarded as pullback of a function in $C^\infty(T^*\mathcal{B})$.

In order to solve the Gauss constraint (26), we shall comment on its geometrical meaning. Since \mathcal{G} is not necessarily connected for a general principal G -bundle P on the 3-manifold M , only the behaviour of physical states under gauge transformations belonging to the connected component of the identity \mathcal{G}_0 of \mathcal{G} is controlled by the Gauss constraint. Here the gauge group \mathcal{G} appears in the exact sequence

$$1 \rightarrow \mathcal{G}_0 \rightarrow \mathcal{G} \rightarrow \pi_0(\mathcal{G}) \rightarrow 1, \quad (28)$$

where $\pi_0(\mathcal{G})$ denotes the group of components of \mathcal{G} . Correspondingly, one has the principal \mathcal{G}_0 bundle $\mathcal{A} \xrightarrow{\pi'} \tilde{\mathcal{M}} = \mathcal{A}/\mathcal{G}_0$, where $\tilde{\mathcal{M}} \xrightarrow{\tilde{\pi}} \mathcal{M}$ itself has the structure of a principal $\pi_0(\mathcal{G})$ bundle over the gauge orbit space \mathcal{M} . Since $\pi_1(\mathcal{M}) = \pi_0(\mathcal{G})$, $\tilde{\mathcal{M}}$ is the simply connected covering space of \mathcal{M} .

There is a natural *LieG* action on \mathcal{L}_0 induced by the horizontally (with respect to ∇^ε) lifted fundamental vector fields Z_ξ . If the restrictions of \mathcal{L}_0 along the \mathcal{G}_0 orbits in \mathcal{A} have trivial holonomy, the infinitesimal action can be extended to an action of \mathcal{G}_0 on \mathcal{L}_0 . This obstruction is a manifestation of the quantization condition in the Dirac approach.

Let $j_A: \mathcal{G} \rightarrow \mathcal{A}$, $j_A(u) = u^*A$ be the natural inclusion of the fiber then we define the functional $F'(A, u, c) := \exp(2\pi i \int_c j_A^* \varepsilon)$ for $(A, u) \in \mathcal{A} \times \mathcal{G}_0$ where c is a path in \mathcal{G}_0 with $c(0) = id_{\mathcal{G}}$ and $c(1) = u$. It is evident that the functional F' is independent of the choice of the path c if and only if the cohomology class $[j_A^* \varepsilon]$ belongs to $H^1(\mathcal{G}_0, \mathbb{Z})$ which implies that the holonomy of $j_A^* \mathcal{L}_0$ is trivial. In that case we write $F(A, u) = F'(A, u, c)$ and the \mathcal{G}_0 action on \mathcal{L}_0 is given by $(A, z) \mapsto (u^*A, F(A, u)z)$. This action would be well defined since $F(u_1^*A, u_2)F(A, u_1) = F(A, u_1u_2)$.

In order to show that the quantization obstruction is trivially fulfilled in the present case, let us consider the following one form on \mathcal{A}

$$\kappa = \frac{\theta}{16\pi^2} \left(\int_M Q(\Omega_\varphi)^{(1,3)} - \int_\Sigma (\hat{i} \times id)^* TQ(\omega, \eta)^{(1,2)} \right). \quad (29)$$

Using (9c,g) it is easy to prove that $R_\omega = d_{\mathcal{A}}\kappa$. Since $Q(\Omega_\omega)^{(2,2)}$ is horizontal, one concludes from (7) and (10) that $i_{Z_\xi}\kappa = 0$ and hence κ descends to a form $\bar{\kappa} \in \Omega^1(\mathcal{M}, \mathbb{R})$ such that $\overline{R_\omega} = d_{\mathcal{M}}\bar{\kappa}$. Hence we have proven

PROPOSITION 2. *The De Rham class of $\overline{R_\omega}$ in $H^2(\mathcal{M}, \mathbb{R})$ is trivial.*

It follows from (10), (25) and (29) that

$$j_A^* \varepsilon = -d_{\mathcal{G}_0} j_A^* \frac{\theta}{16\pi^2} \int_M TQ(\varphi, a)^{(0,3)} \quad (30)$$

is an exact one form on \mathcal{G}_0 and since $TQ(\varphi, a)^{(0,3)} = TQ(A, a)$, one obtains

$$F(A, u) = \exp \left(-\frac{i\theta}{8\pi} \left(\int_M TQ(u^*A, a) - \int_M TQ(A, a) \right) \right). \quad (31)$$

If we factorize by this \mathcal{G}_0 action, we get the line bundle $\mathcal{L}_F = \mathcal{A} \times_F \mathbb{C}$ on $\tilde{\mathcal{M}}$. Let $\gamma(t)$ be a path in \mathcal{A} with $\gamma(0) = A$ and $u \in \mathcal{G}_0$ then ε satisfies

$$\varepsilon_{u^*A} \left(\frac{d}{dt} \Big|_{t=0} u^* \gamma(t) \right) - \varepsilon_A \left(\frac{d}{dt} \Big|_{t=0} \gamma(t) \right) = \frac{d}{dt} \Big|_{t=0} F(A, u)^{-1} F(\gamma(t), u), \quad (32)$$

which is the necessary and sufficient condition for ε descending to a well defined connection $\bar{\varepsilon}$ on the line bundle \mathcal{L}_F . Since $(\mathcal{L}_F, \bar{\varepsilon})$ is the unique (up to bundle equivalence) line bundle on $\tilde{\mathcal{M}}$ with curvature $-i\tilde{\pi}^* \overline{R_\omega}$, it is trivializable and therefore no quantization condition must be imposed on the coupling parameter θ to obtain non-trivial solutions of (26).

Let $Aut(\mathcal{L}_F, \bar{\varepsilon})$ denote the group of bundle automorphisms on \mathcal{L}_F leaving the connection $\bar{\varepsilon}$ invariant and let $r: \tilde{\mathcal{M}} \times \pi_0(\mathcal{G}) \rightarrow \tilde{\mathcal{M}}$ be the induced principal action

of $\pi_0(\mathcal{G})$ on $\tilde{\mathcal{M}}$. Since \mathcal{L}_F is trivializable, there exists a lifting [16] $\nu: \pi_0(\mathcal{G}) \rightarrow \text{Aut}(\mathcal{L}_F, \tilde{\varepsilon})$ of r . The space of orbits $\mathcal{L}_{F,\nu} := \mathcal{L}_F/\nu$ of ν on \mathcal{L}_F is a line bundle on \mathcal{M} with induced connection $\tilde{\varepsilon}$ and curvature $-i\overline{R_\omega} \in \Omega^2(\mathcal{M}, \mathbb{R})$.

According to Prop. 2, the quantum line bundle $\mathcal{L}_{F,\nu}$ has vanishing real Chern class and therefore its topological type is classified by the image of the Bockstein operator $\delta^*: H^1(\mathcal{M}, \mathbb{R}/\mathbb{Z}) \rightarrow H^2(\mathcal{M}, \mathbb{Z})$. Notice that $H^1(\mathcal{M}, \mathbb{R}/\mathbb{Z}) \cong \text{Hom}(\pi_0(\mathcal{G}), \mathbb{R}/\mathbb{Z})$. In fact, $\mathcal{L}_{F,\nu}$ also depends on the chosen background field $a \in \mathcal{A}$ but it can be easily shown that the line bundles corresponding to different background fields are isomorphic.

PROPOSITION 3. *The space of physical admissible states is isomorphic to the space of sections of the line bundle $\mathcal{L}_{F,\nu}$ on \mathcal{M} .*

Although the wave functionals can be chosen to be \mathcal{G}_0 invariant, a non-trivial lift ν represents an obstruction to extend them to the whole \mathcal{A} in a \mathcal{G} invariant way. If the abelization of $\pi_1(\mathcal{M}) = \pi_0(\mathcal{G})$, namely $H_1(\mathcal{M}, \mathbb{Z})$, is torsionless, then $\mathcal{L}_{F,\nu}$ will be trivializable.

Let us remark that for suitable manifolds the two form $\int_\Sigma Q(\Omega_{\tilde{\omega}})^{(2,2)}$ belongs to the generating class in $H^2(\mathcal{N}, \mathbb{Z})$ [6]. Hence the θ parameter would be quantized if and only if the quantum line bundle $\mathcal{L}_{F,\nu}$ was the pullback of a line bundle on \mathcal{N} . However, according to the principles of geometric quantization, there is no reason that this may be the case.

The isomorphism class of $\mathcal{L}_{F,\nu}$ (as a unitary line bundle with connection) is not unique, giving rise to inequivalent quantum theories, because the lift ν is determined only up to elements of $\text{Hom}(\pi_0(\mathcal{G}), \mathbb{R}/\mathbb{Z})$. Generally, the possible quantum line bundles are of the form $\mathcal{L}_{F,\nu} \otimes \mathcal{V}$, where \mathcal{V} is a flat unitary line bundle on \mathcal{M} .

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